Raritan Valley Math Group Spring Contest Solutions

Series A

1. Either by long division or by the observations that $1 = 0.\overline{9999} = 9999 \cdot 0.\overline{0001}$, we see that $\frac{1234}{9999} = 0.\overline{1234}$. Hence the sum of the first 2016 entries is $\frac{2016}{4} \cdot (1+2+3+4) = 5040$. The 2017th digit is 1, so the sum of the first 2017 digits after the decimal place is $5040 + 1 = \boxed{5041}$

2. The last two digits of the answer to the previous question are 41. Thus, we wish to compute $(-11)+(-10)+(-9)+...+41 = 12+13+14+...+41$. This sum has $41-12+1 = 30$ numbers whose average is $\frac{12+41}{2} = \frac{53}{2}$. Thus, the desired sum is $\frac{30 \cdot 53}{2} = 15 \cdot 53 = \boxed{795}$.

3. The last two digits of the answer to the previous question are 95. Now $p(95) = 89$. So we seek the sum $(3-2)(3+2)+(5-3)(5+3)+...+(89-83)(89+83) = 3^2-2^2+5^2-3^2+7^2-5^2+...+89^2-83^2$. This telescopes; in other words, all the terms cancel except $89^2 - 2^2 = \boxed{7917}$

4. The last two digits of the answer to the previous question are 17. Thus, we are to compute $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + ... + 15 \cdot 16 \cdot 17$. One approach is to use the hockey stick identity which states that

$$\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1}$$

   The sum we desire is

$$3! \sum_{m=3}^{17} \binom{m}{3} = 3! \binom{18}{4} = \boxed{18360}$$

5. The last two digits of the answer to the previous question are 60. Let the terms of the sequence be given by a_n with $a_1 = 1$. Then $a_{n+2} = a_{n+1} + a_n$.

Rewriting, we get

$$\begin{aligned} a_1 &= a_3 - a_2 \\ a_2 &= a_4 - a_3 \\ &\dots \\ a_m &= a_{m+2} - a_{m+1} \end{aligned}$$

Summing the left hand and right hands sides (and noticing that the right hand side telescopes), we get

$$\sum_{k=1}^{m} a_k = a_{m+2} - a_2$$

Thus we need to compute two more terms of the sequence. The next element is $574916761 + 930234860 = 1505151621$. The following element is $930234860 + 1505151621 = 2435386481$. Thus the desired sum is $2435386481 - 5 = \boxed{2435386476}$.

Series B

1. We only need to consider the last digit so the problem becomes finding the last digit of 7^{2017}. Note that 7^4 ends in 1 (or $7^4 \equiv 1 \mod 10$). As 2016 is divisible by 4, $7^{2017} \equiv 7^{2016} \cdot 7 \equiv 7 \mod 10$. Therefore the last digit is $\boxed{7}$.

2. Note that 8! is divisible by 9 so $(8!)^3$ is divisible by 9. Hence the sum of the digits must be divisible by 9 as well. The sum of the known digits is 49 and the next multiple of 9 is 54 so, the unknown digit must be $54 - 49 = \boxed{5}$.

3. We need to count the powers of 10 in $(28!)^3$. There are, however, more powers of 2 than powers of 5 is 28!, so we need only count powers of 5. There is a power of 5 for every (postive) multiple of 5 less than 28 and another power of 5 for every(positive) multiple of 25 less than 28; hence, there are 6 powers of 5 in 28! and $6 \cdot 3 = 18$ powers of 5 in $(28!)^3$. Therefore, $(28!)^3$ ends in $\boxed{18}$ zeros.

4. Let x be such a number with every positive power having the same last two digits as x. Then we must have $x^2 \equiv x \mod 100$. In other words, $x^2 - x \equiv 0 \mod 100$, so $x(x-1) \equiv 0 \mod 100$. Thus we desire solutions to $x \equiv 0, 1 \mod 4$ and $x \equiv 0, 1 \mod 25$. Considering solutions to the latter yields $\{0, 1, 25, 26, 50, 51, 75, 76\}$. Of these, only $\{0, 1, 25, 76\}$ are solutions to the first equation. The sum of the elements in this set is $\boxed{102}$.

5. As we're seeking powers with even tens digits, we can initially confine our search modulus 20. Clearly, we can rule out 10 through 19. It's probably easiest and quickest to manually inspect each case and arrive at the six solutions in the set $\{0, 1, 3, 5, 7, 9\}$. The sum of the elements in this set is 25. The sum of the possibilities less than 100 would be

$$\sum_{k=0}^{4}(25 + 6 \cdot 20k) = 125 + 120 \cdot 10 = \boxed{1325}$$

Series C

1. Let $X = (1, 1)$. Then $f(X) = XA^2 + XB^2 + XC^2 = (1-0)^2 + (1-0)^2 + (1-3)^2 + (1-0)^2 + (1-0)^2 + (1-4)^2 = \boxed{17}$ by the Pythagorean Distance Formula.

2. $f(A) = 3^2 + 4^2 = 25$, $f(B) = 3^2 + 3^2 + 4^2 = 34$, and $f(C) = 4^2 + 3^2 + 4^2 = 41$. Therefore, f has its largest value at $\boxed{C}$ out of the set of vertices.

3. Note that if $X = (x, y)$, $f(X) = x^2 + y^2 + (x-3)^2 + y^2 + x^2 + (y-4)^2 = 3x^2 - 6x + 3y^2 - 8y + 25$. By completing the squares, we get $f((x, y)) = 3(x-1)^2 + 3(y - \frac{4}{3})^2 + 25 - 3 - \frac{3 \cdot 16}{9} = 3(x-1)^2 + 3(y - \frac{4}{3})^2 + \frac{50}{3}$. Therefore, f has its minimum value at $\boxed{(1, \frac{4}{3})}$.

4. From the previous question, we saw that $f((x, y)) = 3(x-1)^2 + 3(y - \frac{4}{3})^2 + \frac{50}{3}$. Hence, the level curves of f are circles centered at $P = (1, \frac{4}{3})$. At the vertices, f evaulates to 25, 34, and 41. The distance squared from P to $\overleftrightarrow{AB}$ is $(\frac{4}{3})^2 = \frac{16}{9}$. The distance squared from P to $\overleftrightarrow{AC}$ is 1. And the distance squared from P to $\overleftrightarrow{BC}$ is given by the square of the distance formula between a line and a point:

$$\frac{(ax_0 + by_0 + c)^2}{a^2 + b^2}$$

where the line is given by $ax + by + c = 0$ and the point is (x_0, y_0). In our case, the line between B and C is $4x + 3y - 12 = 0$ so this evaluates to

$$\frac{(4 \cdot 1 + 3 \cdot \frac{4}{3} - 12)^2}{4^2 + 3^2} = \frac{16}{25}$$

. Hence, we can choose any value between $\frac{16}{9}$ and 25 and f at that value will have two solutions per side of the triangle giving $\boxed{6}$ points.

5. Notice that when f is written out as a function of (x, y) that the quadratic in x is dependent only on the first coordinates of the points and the quadratic in y is dependent only on the second coordinates of the points. With this observation, we can make f_S and f_T identical by selecting (a, b) as $(2017, 1)$.

Series D

1. $\frac{1}{4}$ is the root of $4x-1$. $-\frac{5}{6}$ is the root of $6x+5$. The product of these two polynomials is $\boxed{24x^2-14x-5}$

2. For a quadratic to have exactly one real root, it must be a repeated root r. We must have $(x-r)^2 = x^2+ax+2017$ so $r^2 = 2017$ and $r = \pm\sqrt{2017}$ meaning that $a = \mp 2\sqrt{2017}$. As $a > 0$, $a = \boxed{2\sqrt{2017}}$.

3. The roots must be complex conjugates. The sum of the roots then is $\frac{8}{3}$ and the product is $\frac{5}{9}$. A polynomial with the required roots is: $x^2 - \frac{8}{3}x + \frac{5}{9}$. Clearing the fractions to arrive at the desired polynomial, $\boxed{9x^2-24x+5}$.

4. Let r, s, t be the roots of the polynomial. Then $r+s+t = 0$, $rs+rt+st = \frac{1}{2}$, and $rst = -\frac{7}{2}$. We seek $r^3+s^3+t^3$. Note that

   $$(r+s+t)(rs+rt+st) = r^2s+r^2t+s^2r+s^2t+t^2r+t^2s+3rst$$

   Cubing the sum of the roots yields

   $$\begin{aligned}
   0 &= (r+s+t)^3 \\
   &= r^3+s^3+t^3+3(r^2s+r^2t+s^2r+s^2t+t^2r+t^2s)+6rst \\
   &= r^3+s^3+t^3+3(r^2s+r^2t+s^2r+s^2t+t^2r+t^2s+3rst)-3rst \\
   &= r^3+s^3+t^3+3(r+s+t)(rs+rt+st)-3rst \\
   &= r^3+s^3+t^3-3(-\frac{7}{2}) \\
   &= r^3+s^3+t^3+\frac{21}{2}
   \end{aligned}$$

   Therefore the sum of the cubes of the roots is $\boxed{-\frac{21}{2}}$.

5. Let $a = r^{\frac{2}{3}} + r^{-\frac{2}{3}}$. Then $a^2 = r^{\frac{4}{3}} + r^{-\frac{4}{3}} + 2$, so $a^2 - 2 = r^{\frac{4}{3}} + r^{-\frac{4}{3}}$. Hence, $a(a^2-2) = r^2 + r^{-2} + r^{\frac{2}{3}} + r^{-\frac{2}{3}} = r^2 + r^{-2} + a$ so that $a^3 - 3a = r^2 + r^{-2}$. As $r^2 - 14r - 1 = 0$ and $r > 0$, we have $r - 14 - r^{-1} = 0$ or $r - r^{-1} = 14$ Squaring, we get $r^2 + r^{-2} - 2 = 196$ or $r^2 + r^{-2} = 198$. Hence, $a^3 - 3a - 198 = 0$. Factoring, we get $(a-6)(a^2+6a+33) = 0$. $a = 6$ is the only real root since the discriminant of $a^2+6a+33$ is $6^2 - 4 \cdot 33 < 0$. Hence $r^{\frac{2}{3}} + r^{-\frac{2}{3}} = \boxed{6}$

Series E

1. Without loss of generality, suppose a boy leaves first. Then two girls are adjacent. If one of those two girls leaves, then the boys and girls

will alternate; otherwise, they will not. Thus, the desired probability is $\frac{2}{4} = \boxed{\dfrac{1}{2}}$.

2. After one boy leaves, two girls are adjacent. If the second boy leaving is adjacent to the two adjacent girls, then there will be three girls in a row; otherwise, not. There are two boys adjacent to the adjacent girls and one who is not so the desired probability is $\boxed{\dfrac{2}{3}}$.

3. Let's number the chairs clockwise from 1 to 8. Without loss of generality, assume a boy takes chair 8. Now the remaining boys must be in chairs 2, 4, and 6. Also, there are $\binom{7}{3} = 35$ ways of placing three boys in the 7 chairs. Hence, the desired probability is $\boxed{\dfrac{1}{35}}$.

4. As before number the chairs clockwise from 1 to 10 and assume without loss of generality that the boy from family 5 sits in chair 10. Then there are 9! possible arrangements for the remaining boys and girls. For the boys and girls to alternate, boys from families 1 to 4 must sit in chairs 2, 4, 6, and 8. There are 4! ways for this to occur. Now we must place the girls. Now fix the boys in the even numbered chairs so that the boy from family n sits in chair $2n$. To count the number of ways to place the girls, note that there are three allowable positions per girl so that she is not next to her brother. Going in a clockwise manner, label these positions as $-1, 0, 1$. We may construct a sequence of five numbers starting with the placement of the girl from family 1 to the girl from family 5. The conditions on this sequence are: The number immediately to the right of another number can't be exactly one smaller (otherwise two girls will occupy the same position). Also, the number two to the right can't be exactly two smaller. These conditions, of course, loop back from the end back to the beginning of the sequence. With these conditions, we have the following possible sequences:

   (a) -1, -1, -1, -1, -1
   (b) 0, 0, 0, 0, 0
   (c) 1, 1, 1, 1, 1
   (d) -1, 0, 0, 0, 1
   (e) 0, 0, 0, 1, -1
   (f) 0, 0, 1, -1, 0
   (g) 0, 1, -1, 0, 0
   (h) 1, -1, 0, 0, 0
   (i) -1, 0, 1, -1, 1
   (j) 0, 1, -1, 1, -1

(k) 1, -1, 1, -1, 0

(l) -1, 1, -1, 0, 1

(m) 1, -1, 0, 1, -1

Hence, there are 13 possible arrangements for each arrangement of boys. Therefore the probability of arriving at such an arrangement is

$$\frac{13 \cdot 4!}{9!} = \frac{13}{5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} = \boxed{\frac{13}{15120}}$$

5. Note that family swaps preserves the sequence of offsets of sister from brother that is listed in previous solution; however, since the position of the boy from family 5 may move, we must consider all rotations of sequences the same. There are 5 such families: $\{a\}, \{b\}, \{c\}, \{d, e, f, g, h\}$ and $\{i, j, k, l, m\}$. Hence, there are $\boxed{5}$ possible arrangements.

Tie breaker

The actual answer is 16,571,522. For large n, one can estimate the sum of the first n primes just by taking the sum of the first n odd integers which is n^2. However, this includes many multiples of 3, so we can instead take the sum of the first $\frac{3n}{2}$ odd integers and subtract off the $\frac{n}{2}$ multiples of three. This gives an estimate of

$$\frac{9n^2}{4} - 3(\frac{n}{2})^2 = \frac{3n^2}{2}$$

Again, this contains many multiples of 5 so this can be refined again, etc.

It turns out that the sum of the first n primes is approximately

$$\frac{1}{2}\ln(n)n^2$$

In our case, this estimate is about 15.5 million.